# Asymptotics of supremum distribution of a Gaussian process over a Weibullian time

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**Abstract**

Let $\{X(t) : t \in [0, \infty)\}$ be a centered Gaussian process with stationary increments and variance function $\sigma_X^2(t)$. We study the exact asymptotics of $\mathbb{P}(\sup_{t\in[0,T]} X(t) > u)$, as $u \to \infty$, where $T$ is an independent of $\{X(t)\}$ nonnegative Weibullian random variable.

As an illustration we work out the asymptotics of supremum distribution of fractional Laplace motion.

**Key words:** exact asymptotics, fractional Laplace motion, Gaussian process.
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## 1 Introduction

The problem of analyzing the asymptotic properties of

$$\mathbb{P}(\sup_{t\in[0,T]} X(t) > u), \quad \text{as} \quad u \to \infty, \tag{1}$$

for a centered Gaussian process with stationary increments $\{X(t)\}$ and deterministic $T > 0$ plays an important role in many fields of applied and theoretical probability.
One of the seminal results in this subject is the exact asymptotics

$$\mathbb{P}(\sup_{t\in[0,T]} X(t) > u) = \mathbb{P}(X(T) > u)(1 + o(1)) \tag{2}$$

as $u \to \infty$, which holds for a wide class of Gaussian processes with stationary increments and convex variance function (see Berman [3]). We refer to [10] for extensions of this result.

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Some recently studied problems in, e.g., queueing theory, risk theory (see [5],[11]) or hydrodynamics (see Section 5), motivate the analysis of (1) for $T$ being an independent of $\{X(t)\}$ nonnegative random variable. In that case the additional variability of $T$ may influence the form of the asymptotics, leading to qualitatively different structures of asymptotics of (1). This was observed in [5], under the scenario that $T$ has regularly varying tail distribution (see also [1]).
In this paper we focus on the exact asymptotics of (1) when $T$ is an independent of $\{X(t)\}$ random variable with asymptotically Weibullian tail distribution. In Theorem 3.1 we find the structural form of the asymptotics that holds for a wide class of Gaussian processes with stationary increments and convex variance function (see assumptions **A1-A3** in Section 2).
Complementary, in Theorem 3.2 we obtain an explicit form of the asymptotics, which appears to be Weibullian.
Additionally, for $\{X(t)\}$ being a fractional Brownian motion, we provide the exact asymptotics of (1) for the whole range of Hurst parameters $H \in (0,1]$. It appears that in the case of $H < 1/2$ (concave variance function) the exact asymptotics takes qualitatively different form then (2).
Finally, in Section 5 we apply the obtained results to the analysis of extremal behavior of *fractional Laplace motion*, see [7, 8].

## 2 Notation and preliminary results

Let $\{X(t) : t \in [0,\infty)\}$ be a centered Gaussian process with stationary increments, a.s. continuous sample paths, $X(0) = 0$ a.s. and variance function $\sigma_X^2(t) := \mathbb{V}\mathrm{ar}(X(t))$. We assume that

**A1** $\sigma_X^2(\cdot) \in C^1([0,\infty))$ is convex;
**A2** $\sigma_X^2(\cdot)$ is regularly varying at $\infty$ with parameter $\alpha_\infty \in (1,2)$;
**A3** there exists $D > 0$ such that $\sigma_X^2(t) \leq Dt^{\alpha_\infty}$ for each $t \geq 0$.

We introduce the following classes of Gaussian processes:

• **fBm**: $X(t) = B_H(t)$ is a fractional Brownian motion with Hurst parameter $H \in (0,1]$, that is a centered Gaussian process with stationary increments and $\sigma_{B_H}^2(t) = t^{2H}$ (note that **A2** is satisfied for $H \in (1/2,1)$);
• **IG**: $X(t) = \int_0^t Z(s)ds$, where $\{Z(t) : t \geq 0\}$ is a centered stationary Gaussian process with covariance function $R(t) = \mathbb{C}\mathrm{ov}(Z(s), Z(s+t))$ which is regularly varying at $\infty$ with

parameter $\alpha_\infty - 2$.

In this paper we analyze the asymptotics of

$$\mathbb{P}(\sup_{t\in[0,T]} X(t) > u), \tag{3}$$

as $u \to \infty$, where $T$ is an independent of $\{X(t)\}$ nonnegative random variable with asymptotically Weibullian tail distribution; that is

$$\mathbb{P}(T > t) = Ct^\gamma \exp(-\beta t^\alpha)(1 + o(1)), \tag{4}$$

as $u \to \infty$, where $\alpha, \beta, C > 0, \gamma \in \mathbb{R}$. We write $T \in \mathcal{W}(\alpha, \beta, \gamma, C)$ if $T$ satisfies (4).
Let us introduce some notation. For given $H \in (0, 1]$, by $\mathcal{H}_H$ we denote the *Pickands's constant* defined by the following limit

$$\mathcal{H}_H = \lim_{T\to\infty} \frac{\mathcal{H}_H(T)}{T},$$

where $\mathcal{H}_H(T) := \mathbb{E}\exp\left(\sup_{t\in[0,T]} \sqrt{2}B_H(t) - t^{2H}\right)$. Moreover, let $\Psi(u) := \mathbb{P}(\mathcal{N} > u)$, where $\mathcal{N}$ denotes the standard normal random variable. $\dot{\sigma}_X(t)$ denotes the first derivative of $\sigma_X(t)$ and $\dot{\sigma}^2_X(t) = 2\sigma_X(t)\dot{\sigma}_X(t)$ is the first derivative of $\sigma^2_X(t)$.
Finally we present a useful lemma, which is also of independent interest.

**Lemma 2.1** *Let $X \in \mathcal{W}(\alpha_1, \beta_1, \gamma_1, C_1)$, $Y \in \mathcal{W}(\alpha_2, \beta_2, \gamma_2, C_2)$ be independent nonnegative random variables. Then $X \cdot Y \in \mathcal{W}(\alpha, \beta, \gamma, C)$ with*
$\alpha = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$,
$\beta = \beta_1^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \beta_2^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} + \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right]$,
$\gamma = \frac{\alpha_1\alpha_2+2\alpha_1\gamma_2+2\alpha_2\gamma_1}{2(\alpha_1+\alpha_2)}$,
$C = \sqrt{2\pi}C_1C_2\frac{1}{\sqrt{\alpha_1+\alpha_2}}(\alpha_1\beta_1)^{\frac{\alpha_2-2\gamma_1+2\gamma_2}{2(\alpha_1+\alpha_2)}}(\alpha_2\beta_2)^{\frac{\alpha_1-2\gamma_2+2\gamma_1}{2(\alpha_1+\alpha_2)}}$.

The proof of Lemma 2.1 is presented in Section 6.1.

# 3 Main results

In this section we present main results of the paper. We begin with the structural form of the analyzed asymptotics (Theorem 3.1), then we present an explicit asymptotic expansion (Theorem 3.2).

**Theorem 3.1** *Let $X(t)$ be a centered Gaussian process with stationary increments and variance function that satisfies **A1** - **A3** and $T \in \mathcal{W}(\alpha,\beta,\gamma,C)$ be an independent of $\{X(t)\}$ nonnegative random variable. Then*

$$\mathbb{P}(\sup_{s\in[0,T]} X(s) > u) \quad = \quad \mathbb{P}(X(T) > u)(1+o(1)) = \mathbb{P}(\sigma_X(T)\cdot\mathcal{N} > u)(1+o(1))$$

*as $u \to \infty$.*

The proof of Theorem 3.1 is presented in Section 6.2.

**Remark 3.1** *The asymptotics obtained in Theorem 3.1 is qualitatively different then the one observed in [5], where it was considered the case of $T$ having regularly varying tail distribution.*

If the variance function of $\{X(t)\}$ is regular enough (in such a way that $\sigma_X(T)$ is asymptotically Weibullian), then the combination of Theorem 3.1 with Lemma 2.1 enables us to get the exact form of the asymptotics.

**Theorem 3.2** *Let $X(t)$ be a centered Gaussian process with stationary increments and variance function that satisfies* **A1** *and $\sigma_X^2(t) = Dt^{\alpha_\infty} + o(t^{\alpha_\infty-\alpha})$ as $t \to \infty$ for $\alpha_\infty \in (1,2)$ and $D > 0$. If $T \in \mathcal{W}(\alpha,\beta,\gamma,C)$ is an independent of $\{X(t)\}$ nonnegative random variable, then*

$$\sup_{s\in[0,T]} X(s) \in \mathcal{W}(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\widetilde{C})$$

*with* $\widetilde{\alpha} = \frac{2\alpha}{\alpha+\alpha_\infty}$, $\widetilde{\beta} = \beta^{\frac{\alpha_\infty}{\alpha+\alpha_\infty}} \left(\frac{D}{2}\right)^{\frac{\alpha}{\alpha+\alpha_\infty}} \left(\left(\frac{\alpha}{\alpha_\infty}\right)^{\frac{\alpha_\infty}{\alpha+\alpha_\infty}} + \left(\frac{\alpha_\infty}{\alpha}\right)^{\frac{\alpha}{\alpha+\alpha_\infty}}\right)$,
$\widetilde{\gamma} = \frac{2\gamma}{\alpha+\alpha_\infty}$, $\widetilde{C} = CD^{-1/\alpha_\infty}\sqrt{\frac{\alpha_\infty}{2(\alpha+\alpha_\infty)}}\left(\frac{\alpha_\infty}{2\alpha\beta}D^{\alpha_\infty/\alpha}\right)^{\frac{\gamma}{\alpha+\alpha_\infty}}$.

The proof of Theorem 3.2 is given in Section 6.3.

Below we apply the obtained asymptotics to **IG** processes. The family of **fBm** is analyzed separately in Section 4. Due to self-similarity of **fBm** we were able to give (an independent of Theorem 3.1) proof that covers the whole range of Hurst parameters $H \in (0,1]$.

**Example 3.1** Let $T \in \mathcal{W}(\alpha,\beta,\gamma,C)$ and $X(t) = \int_0^t Z(s)ds$, where $\{Z(s) : s \geq 0\}$ is a centered stationary Gaussian process with continuous covariance function $R(t)$ such that $R(t) = Dt^{\alpha_\infty-2} + o(t^{\alpha_\infty-2-\alpha})$ as $t \to \infty$ with $\alpha_\infty \in (1,2)$. Following Karamata theorem (see, e.g., Proposition 1.5.8 in Bingham [4] )

$$\sigma_X^2(t) = 2\int_0^t ds \int_0^s R(v)dv = \frac{2D}{\alpha_\infty(\alpha_\infty-1)}t^{\alpha_\infty} + o(t^{\alpha_\infty-\alpha})$$

as $t \to \infty$. Hence, by Theorem 3.2 we have

$$\sup_{t\in[0,T]} X(t) \in \mathcal{W}(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{C})$$

with $\widetilde{\alpha} = \frac{2\alpha}{\alpha+\alpha_\infty}$, $\widetilde{\beta} = \beta^{\frac{\alpha_\infty}{\alpha+\alpha_\infty}} \left(\frac{D}{\alpha_\infty(\alpha_\infty-1)}\right)^{\frac{\alpha}{\alpha+\alpha_\infty}} \left(\left(\frac{\alpha}{\alpha_\infty}\right)^{\frac{\alpha_\infty}{\alpha+\alpha_\infty}} + \left(\frac{\alpha_\infty}{\alpha}\right)^{\frac{\alpha}{\alpha+\alpha_\infty}}\right)$,
$\widetilde{\gamma} = \frac{2\gamma}{\alpha+\alpha_\infty}$, $\widetilde{C} = C\left(\frac{2D}{\alpha_\infty(\alpha_\infty-1)}\right)^{-1/\alpha_\infty} \sqrt{\frac{\alpha_\infty}{2(\alpha+\alpha_\infty)}} \left(\frac{\alpha_\infty}{2\alpha\beta}\left(\frac{2D}{\alpha_\infty(\alpha_\infty-1)}\right)^{\alpha_\infty/\alpha}\right)^{\frac{\gamma}{\alpha+\alpha_\infty}}$.

# 4 The case of fBm

In this section we focus on the exact asymptotics of (3) for $\{X(t)\}$ being an **fBm**. The self-similarity of **fBm**, combined with Lemma 2.1, enables us to provide the following theorem.

**Theorem 4.1** *Let $\{B_H(s) : s \geq 0\}$ be an* **fBm** *with Hurst parameter $H \in (0, 1]$ and $T \in \mathcal{W}(\alpha, \beta, \gamma, C)$ be an independent of $\{B_H(s) : s \geq 0\}$ nonnegative random variable. If*
(i) $H \in (0, 1/2)$, *then*

$$\sup_{s\in[0,T]} B_H(s) \in \mathcal{W}\left(\frac{2\alpha}{2H+\alpha}, \beta_1, \frac{2\alpha - 3\alpha H + 2\gamma}{\alpha + 2H}, C_1\right),$$

(ii) $H = 1/2$, *then*

$$\sup_{s\in[0,T]} B_H(s) \in \mathcal{W}\left(\frac{2\alpha}{2H+\alpha}, \beta_1, \frac{2\gamma}{\alpha + 2H}, 2C_2\right),$$

(iii) $H \in (1/2, 1]$, *then*

$$\sup_{s\in[0,T]} B_H(s) \in \mathcal{W}\left(\frac{2\alpha}{2H+\alpha}, \beta_1, \frac{2\gamma}{\alpha + 2H}, C_2\right),$$

*where*
$\beta_1 = \beta^{\frac{2H}{2H+\alpha}} \left(\frac{1}{2}\left(\frac{\alpha}{H}\right)^{\frac{2H}{2H+\alpha}} + \left(\frac{\alpha}{H}\right)^{-\frac{\alpha}{2H+\alpha}}\right)$,
$C_1 = \mathcal{H}_H \left(\frac{1}{2}\right)^{\frac{1}{2H}} \frac{C}{\sqrt{2H+\alpha}} H^{\frac{\alpha+6H+2\gamma-2}{2\alpha+4H}} (\alpha\beta)^{\frac{1-2H-\gamma}{\alpha+2H}}$,
$C_2 = \frac{C\sqrt{H}}{\sqrt{\alpha+2H}} \left(\frac{H}{\alpha\beta}\right)^{\frac{\gamma}{\alpha+2H}}$.

The following lemma plays an important role in the proof of Theorem 4.1.

**Lemma 4.2** *Let $B_H(\cdot)$ be an* **fBm** *with Hurst parameter $H \in (0, 1]$. If*
(i) $H \in (0, 1/2)$, *then*

$$\sup_{t\in[0,1]} B_H(t) \in \mathcal{W}\left(2, \frac{1}{2}, \frac{1}{H} - 3, \frac{1}{H\sqrt{\pi}} 2^{-\frac{H+1}{2H}}\right),$$

(ii) $H = 1/2$, *then*

$$\sup_{t\in[0,1]} B_H(t) \in \mathcal{W}\left(2, \frac{1}{2}, -1, \frac{2}{\sqrt{2\pi}}\right),$$

(iii) $H \in (1/2, 1]$, *then*

$$\sup_{t\in[0,1]} B_H(t) \in \mathcal{W}\left(2, \frac{1}{2}, -1, \frac{1}{\sqrt{2\pi}}\right).$$

**Proof** Case $H \in (0,1)$ follows by an application of Theorem D3 in [10]. Indeed, by inspection we have that $t_{max} := \max_{t\in[0,1]} \mathbb{V}\mathrm{ar}(B_H(t)) = 1$,

$$1 - \sigma_{B_H}(t + t_{max}) = 1 - \sigma_{B_H}(t+1) = 1 - (t+1)^H = H|t|(1 + o(1)),$$

as $t \to 0-$,

$$1 - \mathbb{C}\mathrm{ov}\left(\frac{B_H(s)}{s^H}, \frac{B_H(t)}{t^H}\right) = 1 - \frac{s^{2H} + t^{2H} - |t-s|^{2H}}{2s^H t^H} = \frac{1}{2}|t-s|^{2H} + o(|t-s|^{2H})$$

as $s, t \to 1-$ and

$$\mathbb{E}(B_H(t) - B_H(s))^2 = |t-s|^{2H}.$$

This (due to Theorem D3 in [10]) implies the asymptotics for $H \in (0,1)$.
The case $H = 1$ follows by the fact that $B_1(t) = \mathcal{N}t$. Hence

$$\mathbb{P}(\sup_{s\in[0,1]} B_1(s) > u) = \mathbb{P}(\sup_{s\in[0,1]} s\mathcal{N} > u) = \mathbb{P}(\mathcal{N} > u) = \frac{1}{\sqrt{2\pi}} u^{-1} \exp\left(-\frac{u^2}{2}\right)(1 + o(1))$$

as $u \to \infty$. This completes the proof. $\square$

**Proof of Theorem 4.1**
Using self-similarity of **fBm** we have

$$\mathbb{P}(\sup_{s\in[0,T]} B_H(s) > u) = \mathbb{P}(T^H \sup_{s\in[0,1]} B_H(s) > u).$$

Notice that $T^H \in \mathcal{W}\left(\frac{\alpha}{H}, \beta, \frac{\gamma}{H}, C\right)$ and (due to Lemma 4.2) $\sup_{s\in[0,1]} B_H(s)$ is asymptotically Weibullian.
Thus all the cases (i), (ii) and (iii) follow by a straightforward application of Lemma 2.1. This completes the proof. $\square$

**Remark 4.1** *Note that if* $\mathbb{P}(T > t) = \exp(-At)$, *then for a standard Brownian motion case, some straightforward calculations give*

$$\mathbb{P}(\sup_{t\in[0,T]} B_{1/2}(t) > u) = \exp(-\sqrt{2A}u)$$

*for each* $u \geq 0$.

## 5 Application to extremes of fractional Laplace motion

In this section we apply Theorem 4.1 to the analysis of the asymptotics of supremum distribution of *fractional Laplace motion* over a deterministic interval.
Following [8] we recall the definition of fractional Laplace motion.
Let $\{\Gamma_t; t \geq 0\}$ be a Gamma process with parameter $\nu > 0$, i.e. is a Lévy process such that the increments $\Gamma_{t+s} - \Gamma_t$ have gamma distributions $\mathcal{G}(s/\nu, 1)$, with density

$$f(x) = \frac{1}{\Gamma(\frac{s}{\nu})} x^{\frac{s}{\nu}-1} \exp(-x),$$

where $\Gamma(\cdot)$ denotes gamma function.
Then, by fractional Laplace motion $\mathbf{fLm}(\sigma, \nu)$ we denote process $\{L_H(t); t \geq 0\}$ defined as follows

$$\{L_H(t); t \geq 0\} \stackrel{d}{=} \{\sigma B_H(\Gamma_t); t \geq 0\}.$$

A standard fractional Laplace motion corresponds to $\sigma = \nu = 1$, and is denoted by **fLm**. We refer to Kozubowski et al. [7, 8] for motivations of interest in analysis of this class of stochastic processes.
Before we present the asymptotics of $\mathbb{P}(\sup_{s \in [0,S]} L_H(s) > u)$, let us observe that for given $S > 0$ we have $\Gamma_S \in \mathcal{W}\left(1, 1, S-1, \frac{1}{\Gamma(S)}\right)$. Indeed applying Karamata's theorem (see, e.g., Proposition 1.5.10 in Bingham [4]).

$$\mathbb{P}(\Gamma_S > u) = \frac{1}{\Gamma(S)} \int_u^\infty x^{S-1} e^{-x} dx = \frac{1}{\Gamma(S)} \int_{e^u}^\infty (\log y)^{S-1} y^{-2} dy = \frac{1}{\Gamma(S)} u^{S-1} e^{-u} (1+o(1))$$

as $u \to \infty$.
In the following proposition we give the exact asymptotics of supremum of **fLm** for $H > 1/2$. Let

$$m_H = \left(\frac{1}{2}\right)^{\frac{1}{2H+1}} \left[ \left(\frac{1}{2H}\right)^{\frac{2H}{2H+1}} + \left(\frac{1}{2H}\right)^{\frac{1}{2H+1}} \right].$$

**Proposition 5.1** *Let $L_H$ be a standard* **fLm**. *If $H > 1/2$, then*

$$\sup_{s \in [0,S]} L_H(s) \in \mathcal{W}\left( \frac{2}{2H+1}, m_H, \frac{2S-2}{1+2H}, \frac{H^{\frac{S+2H}{2+4H}}}{\Gamma(S)\sqrt{1+2H}} \right).$$

**Proof** First we consider the lower bound. We observe that

$$\mathbb{P}(\sup_{s \in [0,S]} B_H(\Gamma_s) > u) \geq \mathbb{P}(B_H(\Gamma_S) > u) = \mathbb{P}((\Gamma_S)^H \mathcal{N} > u).$$

Combining the above with the fact that $(\Gamma_S)^H \in \mathcal{W}\left(\frac{1}{H}, 1, \frac{S-1}{H}, \frac{1}{\Gamma(S)}\right)$, $\mathcal{N} \in \mathcal{W}\left(2, \frac{1}{2}, -1, \frac{1}{\sqrt{2\pi}}\right)$ and Lemma 2.1, we obtain tight asymptotical lower bound.

Now we focus on the upper bound. Using the fact that sample paths of $\Gamma$ process are nondecreasing, we get

$$\mathbb{P}(\sup_{s\in[0,S]} B_H(\Gamma_s) > u) \leq \mathbb{P}(\sup_{s\in[0,\Gamma_S]} B_H(s) > u).$$

In order to complete the proof it suffices to apply (iii) of Theorem 4.1. □

**Remark 5.1** *The case $H \leq 1/2$ should be handled with care. The use of argument presented in the proof of Proposition 5.1 gives*

$$\mathbb{P}(\sup_{s\in[0,S]} L_H(s) > u) \geq \frac{1}{\Gamma(S)\sqrt{1+2H}} H^{\frac{S+2H}{2+4H}} u^{\frac{2S-2}{1+2H}} \exp\left(-m_H u^{\frac{2}{2H+1}}\right)(1+o(1))$$

*as $u \to \infty$, and*

$$\mathbb{P}(\sup_{s\in[0,S]} L_H(s) > u) \leq \frac{1}{\Gamma(S)} 2^{-\frac{1}{2H}} H^{-\frac{2H+S+4}{4H+2}} \mathcal{H}_H u^{\frac{2SH-4H+1}{H(2H+1)}} \exp\left(-m_H u^{\frac{2}{2H+1}}\right)(1+o(1))$$

*as $u \to \infty$. The above leads to the following logarithmic asymptotics for $H \in (0, \frac{1}{2}]$*

$$\frac{\log(\mathbb{P}\sup_{s\in[0,S]} L_H(s) > u)}{u^{\frac{2}{2H+1}}} = -m_H(1+o(1))$$

*as $u \to \infty$.*

*In case $H = \frac{1}{2}$, $S = 1$, due to Remark 4.1, we have*

$$\frac{1}{2}\exp(-\sqrt{2}u) \leq P(\sup_{s\in[0,1]} L_{1/2}(s) > u) \leq \exp(-\sqrt{2}u)$$

*for each $u \geq 0$. We conjecture that the exact asymptotics for $H \leq 1/2$ is influenced by the distribution of jumps of $\Gamma(\cdot)$ process.*

# 6 Proofs

In this section we present detailed proofs of Lemma 2.1, Theorem 3.1 and Theorem 3.2.

## 6.1 Proof of Lemma 2.1

We begin with study of the asymptotic

$$\int_{U(x_0(u))} f(x,u)\exp[S(x,u)]dx$$

as $u \to \infty$ for particular forms of $f(x,u)$ and $S(x,u)$, where $x_0(u)$ denotes the point at which the function $S(x,u)$ of $x$ achieves its maximum over $[0,\infty)$ and

$$U(x_0(u)) = \{x : |x - x_0(u)| \le q(u)|S''_{x,x}(x_0(u),u)|^{-1/2}\}$$

for some suitable chosen function $q(u)$.
The following theorem can be found in , e.g., Fedoryouk [6] (Theorem 2.2).

**Lemma 6.1 (Fedoryouk)** *Suppose that there exists function $q(u) \to \infty$ as $u \to \infty$ such that*

$$S''_{x,x}(x,u) = S''_{x,x}(x_0(u),u)[1 + o(1)] \tag{5}$$

*and*

$$f(x,u) = f(x_0(u),u)[1 + o(1)], \tag{6}$$

*as $u \to \infty$ uniformly for $x \in U(x_0(u))$.*
*Then*

$$\int_{U(x_0(u))} f(x,u)\exp[S(x,u)]dx = \sqrt{-\frac{2\pi}{S''_{x,x}(x_0(u),u)}} f(x_0(u),u)\exp[S(x_0(u),u)](1+o(1))$$

*as $u \to \infty$.*

Lemma 6.1 enables us to get the following exact asymptotics, which will play an important role in further analysis.

**Lemma 6.2** *Let $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2 > 0,\ \gamma \in \mathbb{R}$ and $a(u) = u^{\frac{\alpha_1}{2(\alpha_1+\alpha_2)}}$, $A(u) = u^{\frac{2\alpha_1}{\alpha_1+\alpha_2}}$. Then*

$$\int_{a(u)}^{A(u)} x^\gamma \exp\left(-\frac{\beta_1 u^{\alpha_1}}{x^{\alpha_1}} - \beta_2 x^{\alpha_2}\right) dx = Cu^\delta \exp\left[-\beta_3 u^{\alpha_3}\right](1 + o(1))$$

*as $u \to \infty$, where*

$\alpha_3 = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$, $\beta_3 = \beta_1^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \beta_2^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} + \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right]$,

$\delta = \frac{\alpha_1(-\alpha_2+2\gamma+2)}{2(\alpha_1+\alpha_2)}$,

$C = \sqrt{2\pi}\frac{1}{\sqrt{\alpha_1+\alpha_2}}(\alpha_1\beta_1)^{\frac{-\alpha_2+2\gamma+2}{2(\alpha_1+\alpha_2)}}(\alpha_2\beta_2)^{\frac{-\alpha_1-2\gamma-2}{2(\alpha_1+\alpha_2)}}$.

**Proof** Let $x_0(u) = \left(\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\right)^{\frac{1}{\alpha_1+\alpha_2}} u^{\frac{\alpha_1}{\alpha_1+\alpha_2}}$ and $r(u) = u^{\frac{(1-\varepsilon)\alpha_1}{\alpha_1+\alpha_2}}$ for some $\varepsilon \in (0, \min(\alpha_2/2, 1))$. It is convenient to decompose the analyzed integral in the following way

$$\int_{a(u)}^{A(u)} x^\gamma \exp\left(-\frac{\beta_1 u^{\alpha_1}}{x^{\alpha_1}} - \beta_2 x^{\alpha_2}\right) dx =$$

$$= \int_{a(u)}^{x_0(u)-r(u)} + \int_{x_0(u)-r(u)}^{x_0(u)+r(u)} + \int_{x_0(u)+r(u)}^{A(u)} = I_1 + I_2 + I_3.$$

We begin the proof with showing that

$$I_2 = \int_{x_0(u)-r(u)}^{x_0(u)+r(u)} x^\gamma \exp\left(-\frac{\beta_1 u^{\alpha_1}}{x^{\alpha_1}} - \beta_2 x^{\alpha_2}\right) dx = Cu^\delta \exp\left[-\beta_3 u^{\alpha_3}\right](1+o(1)), \qquad (7)$$

as $u \to \infty$, for $\alpha_3 = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$, $\beta_3 = \beta_1^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \beta_2^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} + \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right]$,
$\delta = \frac{\alpha_1(-\alpha_2+2\gamma+2)}{2(\alpha_1+\alpha_2)}$ and $C = \sqrt{2\pi}\frac{1}{\sqrt{\alpha_1+\alpha_2}}(\alpha_1\beta_1)^{\frac{-\alpha_2+2\gamma+2}{2(\alpha_1+\alpha_2)}}(\alpha_2\beta_2)^{\frac{-\alpha_1-2\gamma-2}{2(\alpha_1+\alpha_2)}}$.
For this we check that assumptions of Lemma 6.1 hold for $S(x,u) := -\frac{\beta_1 u^{\alpha_1}}{x^{\alpha_1}} - \beta_2 x^{\alpha_2}$, $f(x,u) := x^\gamma$ and

$$q(u) := \left((\alpha_1+\alpha_2)(\alpha_1\beta_1)^{\frac{\alpha_2-2}{\alpha_1+\alpha_2}}(\alpha_2\beta_2)^{\frac{\alpha_1+2}{\alpha_1+\alpha_2}}\right)^{-\frac{1}{2}} u^{\frac{\alpha_1\alpha_2}{2(\alpha_1+\alpha_2)} - \frac{\varepsilon\alpha_1}{\alpha_1+\alpha_2}}.$$

Indeed, by inspection, we have that $q(u) \to \infty$. Besides, $S(x,u)$ achieves its maximum at $x_0(u) = \left(\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\right)^{\frac{1}{\alpha_1+\alpha_2}} u^{\frac{\alpha_1}{\alpha_1+\alpha_2}}$ and $U(x_0(u)) = [x_0(u)-r(u), x_0(u)+r(u)]$ with $r(u) = u^{\frac{(1-\varepsilon)\alpha_1}{\alpha_1+\alpha_2}}$.
In order to check (5), we note that

$$S''_{x,x}(x,u) = -\alpha_1(\alpha_1+1)\beta_1 u^{\alpha_1} x^{-\alpha_1-2} - \alpha_2(\alpha_2-1)\beta_2 x^{\alpha_2-2}$$

and

$$S''_{x,x}(x_0(u),u) = -(\alpha_1+\alpha_2)(\alpha_1\beta_1)^{\frac{\alpha_2-2}{\alpha_1+\alpha_2}}(\alpha_2\beta_2)^{\frac{\alpha_1+2}{\alpha_1+\alpha_2}} u^{\frac{\alpha_1(\alpha_2-2)}{\alpha_1+\alpha_2}}.$$

Hence, following mean value theorem, for $|x| \le r(u)$,

$$\begin{aligned} &|S''_{x,x}(x_0(u)+x,u) - S''_{x,x}(x_0(u),u)| = \\ &= xS'''_{x,x,x}(x_0(u)+\theta,u) \le x\left(\mathrm{Const}_1 u^{\alpha_1} x_0(u)^{-\alpha_1-3} + \mathrm{Const}_2 x_0(u)^{\alpha_2-3}\right) \end{aligned}$$

for some $\mathrm{Const}_1, \mathrm{Const}_2 > 0$ and $|\theta| \le r(u)$, which in view of the fact that $r(u) = o(x_0(u))$ as $u \to \infty$, implies (5).
Finally, (6) holds due to the fact that $r(u) = o(x_0(u))$ as $u \to \infty$. Thus, following Lemma 6.1, we get (7).
In order to complete the proof it suffices to show that $I_1, I_3 = o(I_2)$ as $u \to \infty$. Since proofs for $I_1$ and $I_3$ are similar, we focus on the argument that shows $I_1 = o(I_2)$ as $u \to \infty$. Without loss of generality we assume that $\gamma > 0$. Then

$$I_1 \le (x_0(u)-a(u))^\gamma (x_0(u)-r(u)-a(u)) \exp\left(-\frac{\beta_1 u^{\alpha_1}}{(x_0(u)-r(u))^{\alpha_1}} - \beta_2(x_0(u)-r(u))^{\alpha_2}\right),$$

which combined with the fact that (using Taylor's expansion)

$$
\begin{aligned}
&-\frac{\beta_1 u^{\alpha_1}}{(x_0(u)-r(u))^{\alpha_1}} - \beta_2(x_0(u)-r(u))^{\alpha_2} \\
&= -\beta_1 u^{\alpha_1}\left((x_0(u))^{-\alpha_1} + \alpha_1 r(u)(x_0(u))^{-\alpha_1-1} + \frac{1}{2}\alpha_1(\alpha_1+1)(r(u))^2(x_0(u)-\theta r(u))^{-\alpha_1-2}\right) \\
&\quad -\beta_2\left((x_0(u))^{\alpha_2} - \alpha_2 r(u)(x_0(u))^{\alpha_2-1} + \frac{1}{2}\alpha_2(\alpha_2-1)(r(u))^2(x_0(u)+\theta r(u))^{\alpha_2-2}\right) \\
&= -\beta_3 u^{\alpha_3} - \frac{1}{2}(\alpha_1+\alpha_2)(\alpha_1\beta_1)^{\frac{\alpha_2-2}{\alpha_1+\alpha_2}}(\alpha_2\beta_2)^{\frac{\alpha_1+2}{\alpha_1+\alpha_2}} u^{\alpha_1(\alpha_2-2\varepsilon)/(\alpha_1+\alpha_2)}(1+o(1)),
\end{aligned}
$$

as $u \to \infty$, for some $\theta \in [0,1]$, straightforwardly implies $I_1 = o(I_3)$ as $u \to \infty$ (since $\varepsilon < \alpha_2/2$). This completes the proof. □

**Proof of Lemma 2.1**

Let $X \in \mathcal{W}(\alpha_1, \beta_1, \gamma_1, C_1)$ and $Y \in \mathcal{W}(\alpha_2, \beta_2, \gamma_2, C_2)$ be independent nonnegative random variables. Define $a(u) = u^{\frac{\alpha_1}{2(\alpha_1+\alpha_2)}}$, $A(u) = u^{\frac{2\alpha_1}{\alpha_1+\alpha_2}}$ and make the following decomposition

$$
\begin{aligned}
\mathbb{P}(XY > u) &= \int_0^\infty \mathbb{P}\left(X > \frac{u}{y}\right) dF_Y(y) \\
&= \int_0^{a(u)} \mathbb{P}\left(X > \frac{u}{y}\right) dF_Y(y) + \int_{a(u)}^{A(u)} \mathbb{P}\left(X > \frac{u}{y}\right) dF_Y(y) + \int_{A(u)}^\infty \mathbb{P}\left(X > \frac{u}{y}\right) dF_Y(y) \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

We analyze each of the integrals $I_1, I_2, I_3$ separately. In order to short the notation we introduce

$$
\begin{aligned}
h_1(u) &= C_1 u^{\gamma_1}\exp\left(-\beta_1 u^{\alpha_1}\right), \\
h_2(u) &= C_2 u^{\gamma_2}\exp\left(-\beta_2 u^{\alpha_2}\right).
\end{aligned}
$$

**Integral $I_1$**

Since $X \in \mathcal{W}(\alpha_1, \beta_1, \gamma_1, C_1)$, then for given $\varepsilon > 0$ and $u$ large enough, we can bound $I_1$ from above by

$$
\begin{aligned}
\int_0^{a(u)} P\left(X > \frac{u}{y}\right) dF_Y(y) &\le (1+\varepsilon)\int_0^{a(u)} h_1\left(\frac{u}{y}\right) dF_Y(y) \\
&\le (1+\varepsilon) h_1\left(\frac{u}{a(u)}\right) \\
&= (1+\varepsilon) C_1 u^{\frac{\alpha_1+2\alpha_2}{2(\alpha_1+\alpha_2)}\gamma_1}\exp\left(-\beta_1 u^{\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}+\frac{\alpha_1^2}{2(\alpha_1+\alpha_2)}}\right).
\end{aligned}
$$

**Integral $I_3$**

For $u$ sufficiently large, we have

$$
\int_{A(u)}^\infty P\left(X > \frac{u}{y}\right) dF_Y(y) \le \mathbb{P}(Y > A(u)) = C_2 u^{\frac{2\alpha_1\gamma_2}{\alpha_1+\alpha_2}}\exp\left(-\beta_2 u^{\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}}\right)(1+o(1))
$$

as $u \to \infty$.

**Integral $I_2$**

We find upper and lower bound of $I_2$ separately. Using that $X, Y$ are asymptotically Weibullian, we get for sufficiently large $u$

$$\begin{aligned}
&\int_{a(u)}^{A(u)} \mathbb{P}\left(X > \frac{u}{y}\right) dF_Y(y) \geq (1-\varepsilon) \int_{a(u)}^{A(u)} h_1\left(\frac{u}{y}\right) dF_Y(y) \\
\geq\ & (1-\varepsilon) \int_{a(u)}^{A(u)} \frac{\partial}{\partial y}\left[h_1\left(\frac{u}{y}\right)\right] \mathbb{P}(Y > y) dy + (1-\varepsilon) h_1\left(\frac{u}{a(u)}\right) \mathbb{P}(Y > (a(u))) \\
& -(1-\varepsilon) h_1\left(\frac{u}{A(u)}\right) \mathbb{P}(Y > A(u)) \\
\geq\ & (1-\varepsilon)^2 \int_{a(u)}^{A(u)} \frac{\partial}{\partial y}\left[h_1\left(\frac{u}{y}\right)\right] h_2(y) dy + (1-\varepsilon)^2 h_1\left(\frac{u}{a(u)}\right) h_2(a(u)) \\
& -(1-\varepsilon^2) h_1\left(\frac{u}{A(u)}\right) h_2(A(u)) \\
=\ & (1-\varepsilon)^2 I_4 + (1-\varepsilon)^2 R_1 - (1-\varepsilon^2) R_2.
\end{aligned}$$

Analogously, for sufficiently large $u$, we have the upper bound

$$I_2 \leq (1+\varepsilon)^2 I_4 + (1+\varepsilon)^2 R_1 - (1-\varepsilon^2) R_2.$$

Additionally,

$$R_1 = h_1\left(\frac{u}{a(u)}\right) h_2(a(u)) \leq h_1\left(\frac{u}{a(u)}\right) = C_1 u^{\frac{\alpha_1\gamma_1 + 2\alpha_2\gamma_1}{2(\alpha_1+\alpha_2)}} \exp\left(-\beta_1 u^{\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} + \frac{\alpha_1^2}{2(\alpha_1+\alpha_1)}}\right)$$

and

$$R_2 = h_1\left(\frac{u}{A(u)}\right) h_2(A(u)) \leq h_2(A(u)) = C_2 u^{\frac{2\alpha_1\gamma_2}{\alpha_1+\alpha_2}} \exp\left(-\beta_2 u^{\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}}\right).$$

Finally, we find the asymptotics of integral $I_4$

$$\begin{aligned}
I_4 =\ & C_1 C_2 \alpha_1 \alpha_2 u^{\alpha_1+\gamma_1} \int_{a(u)}^{A(u)} y^{-\alpha_1-\gamma_1+\gamma_2-1}\left[\exp\left(\frac{-\beta_1 u^{\alpha_1}}{y^{\alpha_1}} - \beta_2 y^{\alpha_2}\right)\right] dy \\
& -C_1 C_2 \gamma_1 u^{\gamma_1} \int_{a(u)}^{A(u)} y^{-\gamma_1+\gamma_2-1}\left[\exp\left(\frac{-\beta_1 u^{\alpha_1}}{y^{\alpha_1}} - \beta_2 y^{\alpha_2}\right)\right] dy \\
=\ & C_3 u^{\gamma_3} \exp\left(-\beta_3 u^{\alpha_3}\right)\left[1 + \frac{\gamma_1}{\alpha_1+\beta_1}\left(\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} u^{-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}}\right](1+o(1)) \qquad (8) \\
=\ & C_3 u^{\gamma_3} \exp\left(-\beta_3 u^{\alpha_3}\right)(1+o(1))
\end{aligned}$$

as $u \to \infty$, where (8) follows from Lemma 6.2 with
$\alpha_3 = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$, $\beta_3 = \beta_1^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \beta_2^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} + \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right]$, $\gamma_3 = \frac{\alpha_1\alpha_2 + 2\alpha_1\gamma_2 + 2\alpha_2\gamma_1}{2(\alpha_1+\alpha_2)}$,
$C_3 = \sqrt{2\pi} C_1 C_2 \frac{1}{\sqrt{\alpha_1+\alpha_2}} (\alpha_1\beta_1)^{\frac{\alpha_2-2\gamma_1+2\gamma_2}{2(\alpha_1+\alpha_2)}} (\alpha_2\beta_2)^{\frac{\alpha_1-2\gamma_2+2\gamma_1}{2(\alpha_1+\alpha_2)}}$.

Since $I_1, I_2, R_1, R_2 = o((C_3 u^{\gamma_3}\exp(-\beta_3 u^{\alpha_3}))$ as $u\to\infty$, then

$$\mathbb{P}(X\cdot Y>u) = I_4(1+o(1)) = C_3 u^{\gamma_3}\exp\left(-\beta_3 u^{\alpha_3}\right)(1+o(1))$$

as $u\to\infty$. This completes the proof. $\square$

## 6.2 Proof of Theorem 3.1

Let $\tau_1 = \frac{2}{\alpha_\infty+2\alpha}$, $\tau_2 = \frac{4}{2\alpha_\infty+\alpha}$ and $\delta=\delta(u)=\frac{\sigma_X^3(t)}{\dot\sigma_X(t)}2u^{-2}\log^2 u$.
The proof of Theorem 3.1 based on the following three lemmas.

**Lemma 6.3** *Let $X(t)$ be a centered Gaussian process with stationary increments such that conditions **A1 - A3** are satisfied. Then, for sufficiently large $u$,*

$$\mathbb{P}(\sup_{s\in[0,t-\delta]} X(s)>u) \le \Psi\left(\frac{u}{\sigma_X(t)}\right)\exp(-\log^2(u)/2)$$

*uniformly for $t:=t(u)\in[u^{\tau_1},u^{\tau_2}]$.*

**Proof** Let $t:=t(u)\in[u^{\tau_1},u^{\tau_2}]$. Observe that $\sigma_X^2(t)2u^{-2}\log^2(u)\to 0$ uniformly for $t\in[u^{\tau_1},u^{\tau_2}]$ as $u\to\infty$ and $\lim_{t\to\infty}\frac{\sigma_X(t)}{t\dot\sigma_X(t)} = \lim_{t\to\infty}\frac{2\sigma_X^2(t)}{t\dot\sigma_X^2(t)} = \frac{2}{\alpha_\infty}$ (due to (1.11.1) in [4]). Hence

$$\delta(u)=o(t) \text{ as } u\to\infty. \tag{9}$$

Now, for sufficiently large $u$, we make the following decomposition

$$\begin{aligned}
&\mathbb{P}(\sup_{s\in[0,t-\delta]} X(s)>u)\\
&\le \mathbb{P}(\sup_{s\in[0,1]} X(s)>u) + \mathbb{P}\left(\sup_{s\in[1,t-\delta]}\frac{X(s)}{\sigma_X(s)} > \frac{u}{\sigma_X(t-\delta)}\right)\\
&\le \mathbb{P}(\sup_{s\in[0,1]} X(s)>u) + \\
&+ \sum_{k=0}^{\left(\frac{D}{\sigma_X^2(1)}\right)^{\frac{1}{\alpha_\infty}}[t-\delta]} P\left(\sup_{s\in\left[1+\left(\frac{\sigma_X^2(1)}{D}\right)^{\frac{1}{\alpha_\infty}}k,\,1+\left(\frac{\sigma_X^2(1)}{D}\right)^{\frac{1}{\alpha_\infty}}(k+1)\right]}\frac{X(s)}{\sigma_X(s)} > \frac{u}{\sigma_X(t-\delta)}\right).
\end{aligned} \tag{10}$$

According to Borell inequality (see, e.g., Theorem 2.1 in Adler [2]), the first term is bounded by

$$\mathbb{P}(\sup_{s\in[0,1]} X(s)>u) \le \exp\left(-\frac{(u-\mathbb{E}\sup_{s\in[0,1]}X(s))^2}{2}\right)$$

as $u \to \infty$.
In order to find a uniform bound for sum (10), we introduce a centered stationary Gaussian process $\{Z(s) : s \geq 0\}$ with covariance function $\mathbb{C}\mathrm{ov}(Z(s), Z(s+t)) = e^{-t^{\alpha_\infty}}$. The existence of such a process is guaranteed by the fact that $\alpha_\infty < 2$, which implies that the covariance of $Z(\cdot)$ is positively defined; see, e.g., proof of Theorem D.3. in [9].
Due to **A1,A3**, for each $v, w \geq 1$ such that $|v-w| \leq \left(\frac{\sigma_X^2(1)}{D}\right)^{\frac{1}{\alpha_\infty}}$

$$\begin{aligned}
\mathbb{C}\mathrm{ov}\left(\frac{X(v)}{\sigma_X(v)}, \frac{X(w)}{\sigma_X(w)}\right) &= \frac{\sigma_X^2(v) + \sigma_X^2(w) - \sigma_X^2(|v-w|)}{2\sigma_X(v)\sigma_X(w)} \\
&\geq 1 - \frac{\sigma_X^2(|v-w|)}{2\sigma_X(v)\sigma_X(w)} \geq 1 - \frac{\sigma_X^2(|v-w|)}{2\sigma_X^2(1)} \\
&\geq 1 - \frac{D|v-w|^{\alpha_\infty}}{2\sigma_X^2(1)} \\
&\geq \exp\left(-\frac{D}{\sigma_X^2(1)}|v-w|^{\alpha_\infty}\right) \\
&= \mathbb{C}\mathrm{ov}\left(Z\left(\left(\frac{D}{\sigma_X^2(1)}\right)^{\frac{1}{\alpha_\infty}} v\right), Z\left(\left(\frac{D}{\sigma_X^2(1)}\right)^{\frac{1}{\alpha_\infty}} w\right)\right).
\end{aligned}$$

Thus, following Slepian inequality (see, e.g., Theorem C.1 in Piterbarg [10]) each of the summands in (10) can be bounded from the above by

$$\mathbb{P}\left(\sup_{s \in \left[0, \left(\frac{\sigma_X^2(1)}{D}\right)^{\frac{1}{\alpha_\infty}}\right]} Z(s) > \frac{u}{\sigma_X(t-\delta)}\right)$$

This leads to the following upper bound for (10)

$$
\begin{aligned}
&\sum_{k=0}^{\left(\frac{D}{\sigma_X^2(1)}\right)^{\frac{1}{\alpha_\infty}}[t-\delta]} \mathbb{P}\left(\sup_{s\in\left[1+\left(\frac{\sigma_X^2(1)}{D}\right)^{\frac{1}{\alpha_\infty}}k,\,1+\left(\frac{\sigma_X^2(1)}{D}\right)^{\frac{1}{\alpha_\infty}}(k+1)\right]} \frac{X(s)}{\sigma_X(s)} > \frac{u}{\sigma_X(t-\delta)}\right)\\
\le\ & \left(\frac{D}{\sigma_X^2(1)}\right)^{\frac{1}{\alpha_\infty}}[t-\delta]\mathbb{P}\left(\sup_{s\in\left[0,\left(\frac{\sigma_X^2(1)}{D}\right)^{\frac{1}{\alpha_\infty}}\right]} Z(s) > \frac{u}{\sigma_X(t-\delta)}\right)\\
=\ & \mathcal{H}_{\alpha_\infty}[t-\delta]\left(\frac{u}{\sigma_X(t-\delta)}\right)^{\frac{2}{\alpha_\infty}}\Psi\left(\frac{u}{\sigma_X(t-\delta)}\right)(1+o(1)) \qquad (11)\\
=\ & \mathcal{H}_{\alpha_\infty} t\left(\frac{u}{\sigma_X(t)}\right)^{\frac{2}{\alpha_\infty}}\Psi\left(\frac{u}{\sigma_X(t-\delta)}\right)(1+o(1)) \qquad (12)
\end{aligned}
$$

as $u\to\infty$, where (11) follows from Theorem D.2 in Piterbarg [10] and (12) follows from (9). Hence, in order to complete the proof, it suffices to note that

$$
\begin{aligned}
\Psi\left(\frac{u}{\sigma_X(t-\delta)}\right) &\le \frac{2}{\sqrt{2\pi}}\frac{\sigma_X(t-\delta)}{u}\exp\left(-\frac{u^2}{2\sigma_X^2(t-\delta)}\right)\\
&= \frac{1}{\sqrt{2\pi}}\frac{\sigma_X(t-\delta)}{u}\exp\left(-\frac{u^2}{2\sigma_X^2(t)}\right)\exp\left(-\frac{u^2}{2\sigma_X^2(t-\delta)}+\frac{u^2}{2\sigma_X^2(t)}\right)\\
&\le 4\Psi\left(\frac{u}{\sigma_X(t)}\right)\exp\left(-\frac{u^2}{2\sigma_X^2(t-\delta)}+\frac{u^2}{2\sigma_X^2(t)}\right)\\
&\le 4\Psi\left(\frac{u}{\sigma_X(t)}\right)\exp(-\log^2(u)),
\end{aligned}
$$

where the last inequality follows by the following bound ($\theta\in[0,1]$)

$$
\begin{aligned}
\exp\left(-\frac{u^2}{2\sigma_X^2(t-\delta)}+\frac{u^2}{2\sigma_X^2(t)}\right) &= \exp\left(-\frac{u^2(\sigma_X^2(t)-\sigma_X^2(t-\delta))}{2\sigma_X^2(t)\sigma_X^2(t-\delta)}\right)\\
&\le \exp\left(-\frac{u^2(\sigma_X^2(t)-\sigma_X^2(t-\delta))}{2\sigma_X^4(t)}\right)\\
&= \exp\left(-\frac{u^2\delta 2\sigma_X(t-\theta\delta)\dot\sigma_X(t-\theta\delta)}{2\sigma_X^4(t)}\right)\\
&\le \exp\left(-\frac{u^2\delta\sigma_X(t)\dot\sigma_X(t)}{2\sigma^4(t)}\right) \qquad (13)\\
&= \exp(-\log^2(u)) \qquad (14)
\end{aligned}
$$

where (13) is consequence of (9) and of the fact that, by condition **A1**, $\dot{\sigma}_X^2 = 2\sigma_X(t)\dot{\sigma}_X(t)$ is monotone and (in view of (1.11.1) in [4]) regularly varying at $\infty$.
Thus, combining (10) with (12) and (14), for sufficiently large $u$,

$$\begin{aligned}\mathbb{P}(\sup_{s\in[0,t-\delta]} X(s) > u) &\le 4\mathcal{H}_{\alpha_\infty} t \cdot \left(\frac{u}{\sigma_X(t)}\right)^{\frac{2}{\alpha_\infty}} \Psi\left(\frac{u}{\sigma_X(t)}\right) \exp(-\log^2(u))(1+o(1)) \\ &\le \Psi\left(\frac{u}{\sigma_X(t)}\right) \exp(-\log^2(u)/2),\end{aligned}$$

uniformly for $t \in [u^{\tau_1}, u^{\tau_2}]$. This completes the proof. □

**Lemma 6.4** *Let $X(t)$ be a centered Gaussian process with stationary increments such that conditions **A1 - A3** are satisfied. Then, for sufficiently large u,*

$$\mathbb{P}(\sup_{s\in[t-\delta,t]} X(s) > u) \le (1+\varepsilon)\Psi\left(\frac{u}{\sigma_X(t)}\right)$$

*uniformly for $t := t(u) \in [u^{\tau_1}, u^{\tau_2}]$ .*

**Proof** Let $\varepsilon > 0$.Then

$$\begin{aligned}\mathbb{P}(\sup_{s\in[t-\delta,t]} X(s) > u) &= \mathbb{P}\left(\sup_{s\in[t-\delta,t]} \frac{X(s)}{\sigma_X(s)} > \frac{u}{\sigma_X(s)}\right) \\ &\le \mathbb{P}\left(\sup_{s\in[t-\delta,t]} \frac{X(s)}{\sigma_X(s)} > \frac{u}{\sigma_X(t)}\right).\end{aligned}$$

Following argumentation analogous to given in the proof of Lemma 6.3 we get that for each $v, w \in [t-\delta, t]$

$$\begin{aligned}\mathbb{C}\mathrm{ov}\left(\frac{X(v)}{\sigma_X(v)}, \frac{X(w)}{\sigma_X(w)}\right) &= \frac{\sigma_X^2(v) + \sigma_X^2(w) - \sigma_X^2(|v-w|)}{2\sigma_X(v)\sigma_X(w)} \\ &\ge 1 - \frac{\sigma_X^2(|v-w|)}{2\sigma_X(v)\sigma_X(w)} \ge 1 - \frac{\sigma_X^2(|v-w|)}{\sigma_X^2(t)} \\ &\ge 1 - \frac{D|v-w|^{\alpha_\infty}}{\sigma_X^2(t)} \\ &\ge \exp\left(-\frac{2D}{\sigma_X^2(t)}|v-w|^{\alpha_\infty}\right) \\ &= \mathbb{C}\mathrm{ov}\left(Z\left(\left(\frac{2D}{\sigma_X^2(t)}\right)^{\frac{1}{\alpha_\infty}} v\right), Z\left(\left(\frac{2D}{\sigma_X^2(t)}\right)^{\frac{1}{\alpha_\infty}} w\right)\right)\end{aligned}$$

where $\{Z(s) : s \ge 0\}$ is a centred, stationary Gaussian process with covariance function $\mathbb{C}\mathrm{ov}(Z(s), Z(s+t)) = e^{-t^{\alpha_\infty}}$.

Then, from Slepian inequality

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\in[t-\delta,t]}\frac{X(s)}{\sigma_X(s)}>\frac{u}{\sigma_X(t)}\right) &\le \mathbb{P}\left(\sup_{s\in[t-\delta,t]} Z\left(\left(\frac{2D}{\sigma_X^2(t)}\right)^{\frac{1}{\alpha_\infty}} s\right)>\frac{u}{\sigma_X(t)}\right)\\
&= \mathbb{P}\left(\sup_{s\in[0,\delta]} Z\left(\left(\frac{2D}{\sigma_X^2(t)}\right)^{\frac{1}{\alpha_\infty}} s\right)>\frac{u}{\sigma_X(t)}\right)\\
&= \mathbb{P}\left(\sup_{s\in\left[0\,,\,(2D)^{\frac{1}{\alpha_\infty}}u^{\frac{2}{\alpha_\infty}}\delta(\sigma_X(t))^{-\frac{4}{\alpha_\infty}}\left(\frac{u}{\sigma_X(t)}\right)^{-\frac{2}{\alpha_\infty}}\right]} Z(s)>\frac{u}{\sigma_X(t)}\right).
\end{aligned}$$

Observe that for each $\varepsilon_1>0$ there exist $u$ large enough such that $(2D)^{\frac{1}{\alpha_\infty}}u^{\frac{2}{\alpha_\infty}}\delta(\sigma_X(t))^{-\frac{4}{\alpha_\infty}}\le\varepsilon_1$ uniformly for $t\in[u^{\tau_1},u^{\tau_2}]$, which combined with Theorem D.1 in [10], implies the following chain of bounds

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\in\left[0\,,\,(2D)^{\frac{1}{\alpha_\infty}}u^{\frac{2}{\alpha_\infty}}\delta(\sigma_X(t))^{-\frac{4}{\alpha_\infty}}\left(\frac{u}{\sigma_X(t)}\right)^{-\frac{2}{\alpha_\infty}}\right]} Z(s)>\frac{u}{\sigma_X(t)}\right)\\
&\le \mathbb{P}\left(\sup_{s\in\left[0\,,\,\varepsilon_1\left(\frac{u}{\sigma_X(t)}\right)^{-\frac{2}{\alpha_\infty}}\right]} Z(s)>\frac{u}{\sigma_X(t)}\right)\\
&\le (1+\varepsilon_1)\mathcal{H}_{\alpha_\infty}(\varepsilon_1)\Psi\left(\frac{u}{\sigma_X(t)}\right)\le(1+\varepsilon)\Psi\left(\frac{u}{\sigma_X(t)}\right)
\end{aligned}$$

where the last inequality is due to the fact that $\mathcal{H}_{\alpha_\infty}(t)\to 1$ as $t\to 0$.
This completes the proof. $\square$

**Proof of Theorem 3.1.** We find lower and upper bound separately.
**Lower bound.**
It is obvious that

$$\mathbb{P}(\sup_{s\in[0,T]} X(s)>u)\ge\mathbb{P}(X(T)>u).$$

**Upper bound.**
We have

$$\begin{aligned}
\mathbb{P}(\sup_{s\in[0,T]} X(s) > u) &\leq \int_0^{u^{\tau_1}} \mathbb{P}(\sup_{s\in[0,t]} X(s) > u) dF_T(t) \\
&+ \int_{u^{\tau_1}}^{u^{\tau_2}} \mathbb{P}(\sup_{s\in[0,t-\delta]} X(s) > u) dF_T(t) \\
&+ \int_{u^{\tau_1}}^{u^{\tau_2}} \mathbb{P}(\sup_{s\in[t-\delta,t]} X(s) > u) dF_T(t) \\
&+ \int_{u^{\tau_2}}^{\infty} \mathbb{P}(\sup_{s\in[0,t]} X(s) > u) dF_T(t) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Now we investigate asymptotic behavior of each of the integrals.
**Integral $I_1$**

$$\begin{aligned}
\int_0^{u^{\tau_1}} \mathbb{P}(\sup_{s\in[0,t]} X(s) > u) dF_T(t) &\leq \int_0^{u^{\tau_1}} \mathbb{P}(\sup_{s\in[0,u^{\tau_1}]} X(s) > u) dF_T(t) \\
&\leq \mathbb{P}(\sup_{s\in[0,u^{\tau_1}]} X(s) > u) \\
&= \mathbb{P}(\sup_{s\in[0,1]} X(s) > u) + \mathbb{P}(\sup_{s\in[1,u^{\tau_1}]} X(s) > u).
\end{aligned}$$

Following argumentation analogous to given in the proof of Lemma 6.3 we obtain asymptotically upper bound for the sum above

$$\mathrm{Const} u^{\tau_1} \left( \frac{u}{\sigma_X(u^{\tau_1})} \right)^{\frac{2}{\alpha_\infty}} \Psi\left( \frac{u}{\sigma(u^{\tau_1})} \right) \leq \exp\left( -u^{\frac{2\alpha}{\alpha+\alpha_\infty}+\varepsilon} \right) (1+o(1)) \tag{15}$$

as $u \to \infty$; for some $\varepsilon > 0$.
**Integral $I_2$**
According to Lemma 6.3, for all $t \in [u^{\tau_1}, u^{\tau_2}]$ and for $u$ large enough

$$\mathbb{P}(\sup_{s\in[0,t-\delta]} X(s) > u) \leq \Psi\left( \frac{u}{\sigma_X(t)} \right) \exp(-\log^2(u)/2).$$

Hence

$$\begin{aligned}
\int_{u^{\tau_1}}^{u^{\tau_2}} \mathbb{P}(\sup_{s\in[0,t-\delta]} X(s) > u) dF_T(t) &\leq \exp(-\log^2(u)/2) \int_{u^{\tau_1}}^{u^{\tau_2}} \Psi\left( \frac{u}{\sigma_X(t)} \right) dF_T(t) \\
&\leq \exp(-\log^2(u)/2) \int_0^{\infty} \Psi\left( \frac{u}{\sigma_X(t)} \right) dF_T(t) \\
&= \exp(-\log^2(u)/2) \mathbb{P}(X(T) > u) = o(\mathbb{P}(X(T) > u)).
\end{aligned} \tag{16}$$

**Integral $I_3$**

Due to Lemma 6.4, for each $\varepsilon > 0$ and $u$ large enough

$$\begin{aligned}
\int_{u^{\tau_1}}^{u^{\tau_2}} \mathbb{P}(\sup_{s\in[t-\delta,t]} X(s) > u) dF_T(t) &\leq (1+\varepsilon)\int_{u^{\tau_1}}^{u^{\tau_2}} \psi\left(\frac{u}{\sigma_X(t)}\right) dF_T(t) \\
&\leq (1+\varepsilon)\int_0^{\infty} \psi\left(\frac{u}{\sigma_X(t)}\right) dF_T(t) \\
&= (1+\varepsilon)\mathbb{P}(X(T) > u). \qquad (17)
\end{aligned}$$

**Integral $I_4$**

$$\begin{aligned}
\int_{u^{\tau_2}}^{\infty} \mathbb{P}(\sup_{s\in[0,t]} X(s) > u) dF_T(t) &\leq \mathbb{P}(T > u^{\tau_2}) \\
&= Cu^{\gamma\tau_2}\exp(-\beta u^{\alpha\tau_2})(1+o(1)) \\
&\leq \exp\left(-u^{\frac{2\alpha}{\alpha+\alpha_\infty}+\varepsilon}\right)(1+o(1)) \qquad (18)
\end{aligned}$$

as $u \to \infty$ , for some $\varepsilon > 0$.

Observe that for each $\epsilon > 0$ and sufficiently large $u$

$$\mathbb{P}(X(T) > u) = \mathbb{P}(\sigma_X(T)\mathcal{N} > u) \geq \mathbb{P}(\sigma_X(T) > u^{\frac{\alpha_\infty}{\alpha+\alpha_\infty}})\mathbb{P}(\mathcal{N} > u^{\frac{\alpha}{\alpha+\alpha_\infty}}) \geq \exp\left(-u^{\frac{2\alpha}{\alpha+\alpha_\infty}+\epsilon}\right).$$

Thus $I_1, I_2, I_4 = o(I_3)$ as $u \to \infty$, which in view of (17) implies that

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) > u\right) \leq (1+\varepsilon)\mathbb{P}(X(T) > u)$$

for each $\varepsilon > 0$ and sufficiently large $u$.

This completes the proof. $\square$

## 6.3 Proof of Theorem 3.2

By a straightforward inspection we observe that $\sigma_X^2(t)$ satisfies **A1-A3**. Thus, in view of Theorem 3.1,

$$\mathbb{P}(\sup_{s\in[0,T]} X(s) > u) = \mathbb{P}(\sigma_X(T)\cdot\mathcal{N} > u)(1+o(1))$$

as $u \to \infty$. Since $\mathcal{N} \in \mathcal{W}(2, 1/2, -1, 1/\sqrt{2\pi})$ then, due to Lemma 2.1, in order to complete the proof it suffices to show that $\sigma_X(T) \in \mathcal{W}\left(\frac{2\alpha}{\alpha_\infty}, \beta D^{\alpha/\alpha_\infty}, \frac{2\gamma}{\alpha_\infty}, CD^{-1/\alpha_\infty}\right)$, which in view of

$$\begin{aligned}
\mathbb{P}(\sigma_X(T) > u) &= \mathbb{P}(T > (\sigma_X)^{-1}(u)) \\
&= C((\sigma_X)^{-1}(u))^{\gamma}\exp\left(-\beta((\sigma_X)^{-1}(u))^{\alpha}\right)(1+o(1))
\end{aligned}$$

and the fact that $(\sigma_X)^{-1}(u) = D^{-1/\alpha_\infty} u^{2/\alpha_\infty}(1+o(1))$, reduces to

$$\exp\left(-\beta((\sigma_X)^{-1}(u))^\alpha\right) = \exp\left(-\frac{\beta}{D^{\alpha/\alpha_\infty}} u^{2\alpha/\alpha_\infty}\right)(1+o(1)) \tag{19}$$

as $u \to \infty$. Let $f(t) := ((\sigma_X)^{-1}(t))^\alpha$. In order to show (19) it suffices to prove that

$$|f(u) - f(\sigma_X(D^{-1/\alpha_\infty} u^{2/\alpha_\infty}))| \to 0, \tag{20}$$

as $u \to \infty$. Due to the mean value theorem

$$|f(u) - f(\sigma_X(D^{-1/\alpha_\infty} u^{2/\alpha_\infty}))| = |u - \sigma_X(D^{-1/\alpha_\infty} u^{2/\alpha_\infty})| \frac{\alpha((\sigma_X)^{-1}(u+\theta))^{\alpha-1}}{\dot{\sigma}_X((\sigma_X)^{-1}(u+\theta))} \tag{21}$$

for $\theta = o(u)$. Following Bingham *et. al.* [4], p. 59, we have that $\frac{\sigma_X(t)}{t\dot{\sigma}_X(t)} = \frac{2\sigma_X^2(t)}{t\dot{\sigma}_X^2(t)} \to \frac{2}{\alpha_\infty}$ as $t \to \infty$, which applied to (21) with $t := (\sigma_X)^{-1}(u+\theta)$ leads to the following asymptotics for (21)

$$\begin{aligned} &\frac{2\alpha}{\alpha_\infty}|u - \sigma_X(D^{-2/\alpha_\infty} u^{2/\alpha_\infty})| \frac{((\sigma_X)^{-1}(u+\theta))^\alpha}{\sigma_X((\sigma_X)^{-1}(u+\theta))}(1+o(1)) = \\ &= \quad \mathrm{Const}|u - \sigma_X(D^{-1/\alpha_\infty} u^{2/\alpha_\infty})| u^{2\alpha/\alpha_\infty} u^{-1}(1+o(1)) \to 0 \end{aligned}$$

as $u \to \infty$, since $|u - \sigma_X(D^{-1/\alpha_\infty} u^{2/\alpha_\infty})| \leq \left|\frac{u^2 - \sigma_X^2(D^{-1/\alpha_\infty} u^{2/\alpha_\infty})}{u}\right| = o(u^{1-2\alpha/\alpha_\infty})$. This completes the proof. □

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